Binomial or not?

Probability problems typically fall into two types: when we can specify the order of events and when we cannot. Consider the two examples below:

- The only M&M we get that is red of the five we draw is the third one.
- Of the five M&Ms we draw, one of them is red.

In the first, the order of events is clear; we know the first, second, fourth, and fifth M&Ms are not red and the third is red. We can compute this probability using just independence. In the second example, the order is not specified and we only know that one M&M is red but do not know when the red M&M was drawn. This second example requires the binomial model (which still assumes independence between each of the *trials*, or draws). When the number of *successes*, where a success is getting a red M&M, is specified but the order is not, this signals us to use the binomial model. In such problems identify the three parameters below:

- n — the number of trials.
- x — the number of successes.
- p — the probability of a single success in one trial.

A *trial* is each individual event (drawing one M&M). A success is defined in the context of the problem and in this example a "success" is drawing a red M&M. If we say the odds of a single M&M being red is given by 0.15, then we can identify the three values for our example from the second bullet: $n = 5, x = 1, p = 0.15$. Once n, x, and p are identified in a problem, the probability of getting exactly x successes in n draws is given by

$$P(\text{exactly } x \text{ successes}) = \frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$$

For the example above, just plug in 5 for n, 1 for x, and 0.15 for p:

$$\frac{5!}{1!(5-1)!}0.15^1(1-0.15)^{5-1} = \frac{5!}{1!4!}0.15^1 0.85^4 = 0.3915$$

where $5! = 5*4*3*2*1 = 120$, $1! = 1$, and $4! = 4*3*2*1 = 24$. The chances of getting exactly one red M&M of the five M&Ms is 0.3915.

Below are a few additional examples:

1. rolling a '1' two times out of five rolls (we'll assume the die is fair: the chance of a '1' is 1/6).
2. 5 of 7 friends passed their physics final exam on the first try.
3. picking exactly one red M&M of six drawn.

If the chance a friend passing his/her exam is 0.9 and the proportion of M&Ms that are red is still 0.15, identify n, x, and p in each example and verify the answers are 0.161, 0.124, and 0.399.

There is one case we haven't yet addressed: what if we want to find the chance of a range of events, like the probability *at least* 5 of 7 friends will pass their physics final? (Here the complement won't help but it is the first option to consider.) So far we have dealt with the case when we are checking for an exact number of successes. In this setup a case, it is useful to break up the probability to individual cases:

$$\begin{aligned} P(\text{at least 5 of 7}) &= P(\text{exactly 5 of 7 } \textit{or} \text{ exactly 6 of 7 } \textit{or} \text{ exactly 7 of 7}) \\ &= P(\text{exactly 5 of 7}) + P(\text{exactly 6 of 7}) + P(\text{exactly 7 of 7}) \end{aligned}$$

Breaking it up in this way is okay since each event is disjoint (it is impossible to have exactly 5 and exactly 6 friends pass). Each of the three probabilities may now be computed using the binomial model. Find n, x, and p for each of the three probabilities. (The last you could actually do without using the binomial model – if you wish to use the binomial model for it, note that 0! is 1.) Verify the final answer for this problem is $0.124 + 0.372 + 0.478 = 0.974$.